THE MAYER-BOLZA PROBLEM OF THE CALCULUS OF VARIATIONS AND THE THEORY OF OPTIMUM SYSTEMS

(ZADACHA MAIERA-BOL'TSA VARIATSIONNOGO ISCHISLENIIA I TEORIIA OPTIMAL'NYKH SISTEM)

PMM Vol.25, No.4, 1961, pp. 668-679

V. A. TROITSKII
(Leningrad)

(Received March 16, 1961)

The Mayer-Bolza problem of the calculus of variations is described in relation to solution of problems of the theory of optimum systems. The necessary conditions for optimization of processes are established and an investigation of optimum states in linear systems is given.

1. Formulation of the problem. Let a system of n ordinary differential equations of the first order be given

$$g_s = \dot{x}_s - f_s(x_1, \ldots, x_n, u_1, \ldots u_m, t) = 0 \qquad (s = 1, \ldots, n) \tag{1.1}$$

with the finite relations

$$\psi_k = \psi_k(u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r < m) \tag{1.2}$$

describing the behavior of a certain mechanical system. Here, $x_1, \ldots, x_n$ are the coordinates of the system, and the quantities $u_1, \ldots, u_m$ will be called the control parameters, according to the common terminology. We shall consider that the state of the system at the initial time $t = t_0$ is given by the relations

$$x_s(t_0) = x_s^\circ \qquad (s = 1, \ldots, n) \tag{1.3}$$

Moreover, we shall require that the coordinates $x_s(T)$ at a certain, not necessarily fixed time $t = T$ be related by the equations

$$\Phi_l = \Phi_l[x_1(T), \ldots, x_n(T), T] = 0 \qquad (l = 1, \ldots, p \leqslant n) \tag{1.4}$$

We formulate the problem of optimization in the following way.

Determine the functions $x_s(t)$ $(s = 1, \ldots, n)$ satisfying Equations (1.1) and the initial conditions (1.3), and determine the control parameters $u_k(t)$ $(k = 1, \ldots, m)$ connected by the relations (1.2) in such a way that the functional

$$J = J\,[x_1(T), \ldots, x_n(T), T] \tag{1.5}$$

assume a stationary value, with the conditions (1.4) being satisfied at the time $t = T$.

This formulation of the problem is different from the general formulation investigated by Pontriagin [3]. The maximum principle provides the necessary condition for the minimum value of the functional. The solution of the Mayer-Bolza problem in a similar formulation had been given in the lectures of L.I. Lur'e, the contents of which have been extensively used in the following arguments.

A trajectory in the $n + m$ dimensional space $x_1, \ldots, x_n, u_1, \ldots, u_n$ which satisfies the conditions formulated above will be called the extremal.

An essential property of the problems of optimization is the existence of limitations imposed on the parameters $u_k(t)$ and, in general, on the coordinates $x_s(t)$. We shall assume now only the existence of limitations on the parameters of control. In this case it is necessary to consider discontinuous parameters $u_k(t)$. Therefore, in the following the functions $x_s(t)$ will be considered to be continuous, and the parameters $u_k(t)$ and the derivatives of the coordinates $x_s(t)$ will be considered to be functions with a finite number of finite discontinuities in the investigated interval $t_0 \leqslant t \leqslant T$.

The formulation given above extends over a large class of optimization problems. Thus, for instance, in the optimization with respect to high-speed performance, the functional J is to be assumed in the form $J = T$, with the conditions

$$\Phi_l = x_l\,(T) - x_l{}^T = 0 \qquad (l = 1, \ldots, n) \tag{1.6}$$

which corresponds to the problem of optimization of the period of transition of the system from the given initial state (1.3) into the state with the coordinates

$$x_s\,(T) = x_s{}^T \qquad (s = 1, \ldots, n) \tag{1.7}$$

The quantities $x_s{}^T$ may be, obviously, equal to zero. In problems of this type the time of transition T is not fixed. If T is prescribed in advance, an arbitrary coordinate $x_\alpha(T)$ may be prescribed at $t = T$ and, thus, $J = x_\alpha(T)$. The remaining coordinates may be considered as being given, i.e.

$$\Phi_l = x_l(T) - x_l^T = 0 \qquad (l = 1, \ldots, \alpha - 1, \alpha + 1, \ldots, n) \tag{1.8}$$

We consider now the classical problem of Mayer [2].

If $J = J[x_1(T), \ldots, x_n(T)]$, then a certain function of the coordinates at a fixed time $t = T$ is optimized. Using the functionals of the type

$$\int_0^T F[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t]\, dt$$

we can reduce the problem to the simpler problem of Lagrange of the calculus of variations [2]. This case is obviously also included in the formulation given above. In fact, introducing a new coordinate $x_{n+1}(t)$ satisfying the equation

$$g_{n+1} = \dot{x}_{n+1} - F[x_1, \ldots, x_n, u_1, \ldots, u_m, t] = 0$$

we are led to the problem of optimization of the value $x_{n+1}(T)$ of this coordinate at the finite time $t = T$. An analogous assumption may be used in order to take into account a condition of the type

$$\int_0^T \varphi[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t]\, dt = c$$

This condition can be written in the form $\Phi_{n+1} = x_{n+1}(T) - c$. Here, the coordinate $x_{n+1}(t)$ satisfies the equation

$$g_{n+1} = \dot{x}_{n+1} - \varphi[x_1, \ldots, x_n, u_1, \ldots, u_m, t] = 0$$

This statement could be considerably generalized.

The examples given here confirm our proposition that a large number of optimization problems may be formulated in the way described above. In this, the forms of the functional J and the conditions (1.4) usually reflect the physical meaning of the optimization problem.

2. Necessary conditions of extremum of the functional J. We construct the expression

$$I = J + \int_{t_0}^T \left\{ \sum_{s=1}^n \lambda_s(t)\, g_s - \sum_{k=1}^r \mu_k(t)\, \psi_k \right\} dt + \sum_{l=1}^p \rho_l \Phi_l \tag{2.1}$$

where $\lambda_s(t)$, $\mu_k(t)$, and ρ_l are undetermined multipliers of Lagrange. Since its right-hand side terms are equal to zero, the conditions of extremum of J and I are identical.

Calculating the variations of the functional I we shall assume that, in the interval $t_0 \leqslant t \leqslant T$, one point $t = t^*$ exists where the control parameters become discontinuous. The existence of several points of this type would only complicate further calculations. The above assumption splits the interval $t_0 \leqslant t \leqslant T$ into two subintervals $t_0 \leqslant t < t^*$ and $t^* < t \leqslant T$, in which $u_k(t)$ are continuous. Accordingly, we shall denote by $x_s^-(t)$, $u_k^-(t)$, $\lambda_s^-(t)$, $\mu_k^-(t)$ the values of the functions defined above in the interval $t_0 \leqslant t < t^*$ and by $u_s^+(t)$, $u_k^+(t)$, $\lambda_s^+(t)$, $\mu_k^+(t)$ the same values in the interval $t^* < t \leqslant T$.

The formulation of the problem indicates that in the calculation of variation of the functional I we should not vary the time t entering explicitly, for instance, in Equations (1.1) and (1.2). Nevertheless, the existence of the limitations of the type (1.4) necessitates the variation of the abscissa of the end T. Therefore, we shall have to make a distinction between "the variation at the end", for instance $\delta x_s^+(T)$, and "the variation of the end", $\Delta x_s^+(T)$. The relations between them can be easily derived:

$$\Delta x_s^+ (T) = \delta x_s^+ (T) + \dot{x}_s^+ (T)\, \delta T \tag{2.2}$$

Thus, for the variation ΔJ we have the expression

$$\Delta J = \sum_{s=1}^{n} \frac{\partial J}{\partial x_s^+(T)}\, \delta x_s^+(T) + \left[\frac{\partial J}{\partial T} + \sum_{s=1}^{n} \frac{\partial J}{\partial x_s^+(T)}\, \dot{x}_s^+(T)\right] \delta T \tag{2.3}$$

An analogous expression can be obtained for $\Delta\Phi_l$.

Similar remarks apply to the variations of functions at the point $t = t^*$ where $u_k(t)$ become discontinuous. Here it is also necessary to make a distinction between "the variation at the point", $\delta x_s^{\pm}(t^*)$, and "the variation of the point", $\Delta x_s^{\pm}(t^*)$. They are related by

$$\Delta x_s^{\pm} (t^*) = \delta x_s^{\pm} (t^*) + \dot{x}_s^{\pm} (t^*)\, \delta t^* \tag{2.4}$$

We can construct now the variation ΔI. Omitting all the intermediate transformations, we write it in the final form

$$\Delta I = \Delta J + \delta \int_{t_0}^{t^*} \left\{\sum_{s=1}^{n} \lambda_s^- g_s^- - \sum_{k=1}^{r} \mu_k^- \psi_k^- \right\} dt + \int_{t^*}^{T} \left\{\sum_{s=1}^{n} \lambda_s^+ g_s^+ - \sum_{k=1}^{r} \mu_k^+ \psi_k^+ \right\} dt +$$

$$+ \Delta \sum_{l=1}^{p} \rho_l \Phi_l = \int_{t_0}^{t^*} \left\{\sum_{s=1}^{n} \delta\lambda_s^- [\dot{x}_s^- - f_s(x_1^-, \ldots, x_n^-, u_1^-, \ldots, u_m^-, t)] - \right.$$

$$\left. - \sum_{k=1}^{r} \delta\mu_k^- \psi_k (u_1^-, \ldots, u_m^-, t)\right\} dt -$$

$$-\int_{t_0}^{t^*}\Big\{\sum_{s=1}^{n}\delta x_s^-\Big[\dot\lambda_s^- + \sum_{\alpha=1}^{n}\frac{\partial f_\alpha}{\partial x_s^-}\lambda_\alpha^-\Big] + \sum_{k=1}^{m}\delta u_k^-\Big[\sum_{s=1}^{n}\lambda_s^-\frac{\partial f_s}{\partial u_k^-} + \sum_{\beta=1}^{r}\mu_\beta^-\frac{\partial \psi_\beta}{\partial u_k^-}\Big]\Big\}dt +$$
$$+\int_{t^*}^{T}\Big\{\sum_{s=1}^{n}\delta\lambda_s^+[\dot x_s^+ - f_s(x_1^+,\ldots,x_n^+,u_1^+,\ldots,u_m^+,t)] -$$
$$-\sum_{k=1}^{r}\delta\mu_k^+\psi_k(u_1^+,\ldots,u_m^+,t)\Big\}dt - \int_{t^*}^{T}\Big\{\sum_{s=1}^{n}\delta x_s^+\Big[\dot\lambda_s^+ + \sum_{\alpha=1}^{n}\frac{\partial f_\alpha}{\partial x_s^+}\lambda_\alpha^+\Big] +$$
$$+\sum_{k=1}^{m}\delta u_k^+\Big[\sum_{s=1}^{n}\lambda_s^+\frac{\partial f_s}{\partial u_k^+} + \sum_{\beta=1}^{r}\mu_\beta^+\frac{\partial\psi_\beta}{\partial u_k^+}\Big]\Big\}dt +$$
$$+\sum_{s=1}^{n}\Big\{\lambda_s^+(T) + \frac{\partial}{\partial x_s^+(T)}\Big[J + \sum_{l=1}^{p}\rho_l\Phi_l\Big]\Big\}\delta x_s^+(T) + \delta T\frac{d}{dT}\Big[J + \sum_{l=1}^{p}\rho_l\Phi_l\Big] +$$
$$+\sum_{s=1}^{n}[\lambda_s^-(t^*) - \lambda_s^+(t^*)]\,\Delta x_s(t^*) - \sum_{s=1}^{n}[\lambda_s^-(t^*)\,\dot x_s^-(t^*) - \lambda_s^+(t^*)\,\dot x_s^+(t^*)]\,\delta t^* \quad (2.5)$$

It has been shown here that the intervals of the integrals may remain unvaried, because the integrands and δt_0 are equal to zero. The components containing $\delta\rho_l$ vanish for the same reason ($\Phi_l = 0$). In the derivation of Expression (2.5) the formulas of the integration by parts were used

$$\int_{t_0}^{t^*}\lambda_s^-\,\delta\dot x_s^-\,dt = \lambda_s^-(t^*)\,\delta x_s^-(t^*) - \int_{t_0}^{t^*}\dot\lambda_s^-(t)\,\delta x_s^-(t)\,dt \quad (2.6)$$

$$\int_{t^*}^{T}\lambda_s^+\,\delta\dot x_s^+\,dt = \lambda_s^+(T)\,\delta x_s^+(T) - \lambda_s^+(t^*)\,\delta x_s^+(t^*) - \int_{t^*}^{T}\dot\lambda_s^+(t)\,\delta x_s^+(t)\,dt \quad (2.7)$$

as well as the relations (2.4) and the conditions of continuity for the functions $x_s(t)$

$$x_s^-(t^*) = x_s^+(t^*), \qquad \Delta x_s^-(t^*) = \Delta x_s^-(t^*) = \Delta x_s(t^*) \quad (2.8)$$

We note now that variations $\delta x_s^{\pm}(t)$, $\delta\lambda_s^{\pm}(t)$ $(s = 1, \ldots, n)$, $\delta\mu_k^{\pm}(t)$ $(k = 1, \ldots, r)$, δt^*, δT, $\Delta x_s(t^*)$ $(s = 1, \ldots, n)$, $2(m - r)$ variations $\delta\mu_k^{\pm}(t)$, and $n - p$ variations $\delta x_s^+(T)$ are independent. Therefore, it is possible to determine $2r$ Lagrangian multipliers $\mu_k^{\pm}(t)$ and p constants ρ_l in such a way that the coefficients of the dependent variations $\delta u_k^{\pm}(t)$ and p variations $\delta x_s^+(T)$ become zero, and to assume the coefficients of the remaining independent variations equal to zero. After this operation we obtain the system of equations

$$\dot{x}_s^\pm - f_s(x_1^\pm, \ldots, x_n^\pm, u_1^\pm, \ldots, u_m^\pm, t) = 0 \qquad (s = 1, \ldots, n) \quad (2.9)$$

$$\psi_k(u_1^\pm, \ldots, u_m^\pm, t) = 0 \qquad (k = 1, \ldots, r) \quad (2.10)$$

coinciding with (1.1) and (1.2), the equations

$$\dot{\lambda}_s^\pm + \sum_{\alpha=1}^{n} \frac{\partial f_\alpha}{\partial x_s^\pm} \lambda_\alpha^\pm = 0 \qquad (s = 1, \ldots, n) \quad (2.11)$$

$$\sum_{s=1}^{n} \lambda_s^\pm \frac{\partial f_s}{\partial u_k^\pm} + \sum_{\beta=1}^{r} \mu_\beta^\pm \frac{\partial \psi_\beta}{\partial u_k^\pm} = 0 \qquad (k = 1, \ldots, m) \quad (2.12)$$

the boundary conditions for the function $\lambda_s(t)$

$$\lambda_s^+(T) + \frac{\partial}{\partial x_s^+(T)} \Big[J + \sum_{l=1}^{p} \rho_l \Phi_l\Big] = 0 \qquad (s = 1, \ldots, n) \quad (2.13)$$

the equality

$$\frac{d}{dT} \Big[J + \sum_{l=1}^{n} \rho_l \Phi_l\Big] = 0 \quad (2.14)$$

and the Erdmann-Weierstrass conditions

$$\lambda_s^-(t^*) = \lambda_s^+(t^*) \quad (s = 1, \ldots, n), \qquad \sum_{s=1}^{n} [\lambda_s^- \dot{x}_s^- - \lambda_s^+ \dot{x}_s^+]_{t=t^*} = 0 \quad (2.15)$$

These relations should be complemented with the initial conditions (1.3), the continuity conditions (2.8) and Equations (1.4).

In this way, in order to determine $4n + 2m + 2r$ functions $x_s^\pm(t)$, $\lambda_s^\pm(t)$, $u_k^\pm(t)$, $\mu_k^\pm(t)$, we have constructed $4n$ differential equations of the first order (2.9) and (2.11), which introduce $4n$ integration constants, $2m$ relations (2.12), and $2r$ relations (2.10). Thus, $4n$ arbitrary constants, p multipliers ρ_l, and the values of t^* and T, altogether $4n + p + 2$ quantities, remain unknown. To determine these quantities we have n initial conditions (1.3), n continuity conditions (2.8), n boundary conditions (2.13), $n + 1$ Erdmann-Weierstrass conditions (2.15), p relations (1.4), and Equation (2.14). Their number is also $4n + p + 2$; therefore, the problem of determining an extremal may be completely solved.

3. Other forms of the above relations. If the Lagrangian function

$$L = \sum_{s=1}^{n} \lambda_s g_s - \sum_{k=1}^{r} \mu_k \psi_k \quad (3.1)$$

is taken into consideration, then Equations (2.11) and (2.12) may be written in the form of ordinary Euler equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_s} - \frac{\partial L}{\partial x_s} = 0 \quad (s = 1, \ldots, n), \qquad \frac{\partial L}{\partial u_k} = 0 \quad (k = 1, \ldots m) \tag{3.2}$$

constructed in terms of this Lagrangian function. Similarly, the equalities

$$\partial L/\partial \lambda_s = 0, \quad (s = 1, \ldots, n), \quad \partial L/\partial \mu_k = 0 \quad (k = 1, \ldots, r) \tag{3.3}$$

may be established, which yield Equations (2.8) and (2.10).

The first n of the Erdmann-Weierstrass conditions (2.15) may be also formulated in the form of the conditions of continuity of the derivatives of the Lagrangian function

$$(\partial L/\partial \dot{x}_s)^-_{t=t^*} = (\partial L/\partial \dot{x}_s)^+_{t=t^*} \tag{3.4}$$

at the points of discontinuity of $u_k(t)$, and the last one of (2.15) may be replaced by the condition of continuity

$$(H_\lambda^-)_{t=t^*} = (H_\lambda^+)_{t=t^*} \tag{3.5}$$

for the function

$$H_\lambda = \sum_{s=1}^{n} \lambda_s \dot{x}_s = \sum_{s=1}^{n} \lambda_s f_s (x_1, \ldots, x_n, u_1, \ldots, u_m, t) \tag{3.6}$$

The function (3.6) is the basis of the maximum principle of Pontriagin in the theory of optimum systems [3,4]. We note here that in the presence of limitations of the type (1.2) optimum processes correspond to a weak extremum of the functional H_λ. This follows from Equations (2.12), which may be constructed in terms of the function

$$H = H_\lambda + H_\mu = \sum_{s=1}^{n} \lambda_s f_s + \sum_{\beta=1}^{r} \mu_\beta \psi_\beta = H_\lambda \tag{3.7}$$

In addition, we note that Equations (2.9) and (2.11) may be written in the form

$$\dot{x}_s = \frac{\partial H}{\partial \lambda_s} = \frac{\partial H_\lambda}{\partial \lambda_s} \qquad \dot{\lambda}_s = -\frac{\partial H}{\partial x_s} = -\frac{\partial H_\lambda}{\partial x_s} \qquad (s = 1, \ldots, n) \tag{3.8}$$

which is used in the derivation of the maximum principle. Equations (2.10) and (2.12) assume a similar form:

$$\partial H/\partial u_k = 0 \quad (k = 1, \ldots, m), \qquad \partial H/\partial \mu_k = 0 \quad (k = 1, \ldots r) \tag{3.9}$$

Consider now the condition (2.14). After certain elementary transformations it can be written in the following form:

$$\frac{\partial}{\partial T}\Big[J + \sum_{l=1}^{p} \rho_l \Phi_l\Big] = H_\lambda \,|_{t=T} = H\,|_{t=T} \tag{3.10}$$

In the case when the functions f_s and ψ_k do not depend explicitly on time t, Equations (2.9) and (2.11) admit the first integral

$$H = h = \text{const}$$

This may be easily derived by considering the expression

$$\frac{dH}{dt} = \sum_{s=1}^{n} \Big(\frac{\partial H}{\partial x_s}\frac{\partial H}{\partial \lambda_s} - \frac{\partial H}{\partial \lambda_s}\frac{\partial H}{\partial x_s}\Big) \equiv 0 \tag{3.11}$$

Therefore, instead of (3.10) we have

$$\frac{\partial}{\partial T}\Big[J + \sum_{l=1}^{p} \rho_l \Phi_l\Big] = h = \text{const} \tag{3.12}$$

Finally, we shall give a somewhat unusual matrix form of the relations (2.9) to (2.15). We introduce [5] the column matrices x and λ of the order n and the column matrices u and μ of the orders m and r, respectively:

$$x = \{x_1, \ldots, x_n\}, \qquad \lambda = \{\lambda_1, \ldots, \lambda_n\}$$
$$u = \{u_1, \ldots, u_m\}, \qquad \mu = \{\mu_1, \ldots, \mu_r\} \tag{3.13}$$

Furthermore, we construct the column matrices f and ψ according to the rules

$$f = \{f_1, \ldots, f_n\}, \qquad \psi = \{\psi_1, \ldots, \psi_r\} \tag{3.14}$$

and the column matrices of the differential operators

$$\frac{\partial}{\partial x} = \Big\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big\}, \qquad \frac{\partial}{\partial u} = \Big\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\Big\} \tag{3.15}$$

of the orders n and m, respectively. Equations (2.9) to (2.12) may now be written in the form

$$\dot{x} - f = 0, \quad \psi = 0, \quad \dot{\lambda} + \frac{\partial}{\partial x} f'\lambda = 0, \quad \frac{\partial}{\partial u}\,[f'\lambda + \psi'\mu] = 0 \tag{3.16}$$

where primes denote the operation of transposing. The initial conditions are represented as the equality

$$x\,(t_0) = x^\circ \tag{3.17}$$

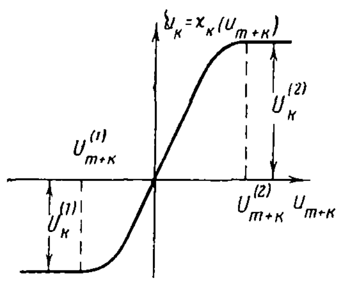
where x° is the column matrix of the initial values x_s°. Similar form is obtained for the continuity conditions (2.8) and (2.15):

$$x^-(t^*) = x^+(t^*), \qquad \lambda^-(t^*) = \lambda^+(t^*) \tag{3.18}$$

The boundary conditions (2.13) assume the form

$$\lambda(T) + \frac{\partial}{\partial x(T)}\{J + \rho'\Phi\} = 0, \quad (\rho = \{\rho_1, \ldots, \rho_p\}, \ \Phi = \{\Phi_1, \ldots, \Phi_p\}) \tag{3.19}$$

where ρ and Φ are column matrices of the order p. The relation (2.14) has the form



$$\frac{d}{dT}[J + \rho'\Phi] = 0 \tag{3.20}$$

and instead of the equality (2.15) we have

$$(\lambda'\dot{x})^-_{t=t^*} - (\lambda'\dot{x})^+_{t=t^*} = 0 \tag{3.21}$$

Note that the following expression may be given for H:

$$H = H_\lambda + H_\mu = \lambda' f + \mu'\psi \tag{3.22}$$

with the alternate formulation (3.5) for the condition (3.21) being preserved.

4. Linear differential equations. We shall consider now the optimization problem for a system of linear differential equations with constant coefficients [6]

$$\dot{x}_s = \sum_{\alpha=1}^{n} b_{s\alpha} x_\alpha + \sum_{\beta=1}^{m'} h_{s\beta} u_\beta \tag{4.1}$$

in terms of the functional J of the general type (1.5) with the limitations

$$U^{(1)}_\beta \leqslant u_\beta \leqslant U^{(2)}_\beta \qquad (\beta = 1, \ldots, m') \tag{4.2}$$

imposed on the control parameters u_β. In order to take into account these limitations we introduce the functions [7,8]

$$u_\beta = \chi_\beta(u_{m'+\beta}) \qquad (\beta = 1, \ldots, m')$$

satisfying the following requirements:

$$\frac{d\chi_\beta}{du_{m'+\beta}} \neq 0, \quad U^{(1)}_\beta < \chi_\beta(u_{m'+\beta}) < U^{(2)}_\beta \quad \text{for } U^{(1)}_{m'+\beta} < u_{m'+\beta} < U^{(2)}_{m'+\beta} \tag{4.3}$$

$$\frac{d\chi_\beta}{du_{m'+\beta}} = 0 \qquad \begin{pmatrix} \chi(u_{m'+\beta}) = U^{(1)}_\beta \ \text{for } u_{m'+\beta} \leqslant U^{(1)}_{m'+\beta} \\ \chi(u_{m'+\beta}) = U^{(2)}_\beta \ \text{for } u_{m'+\beta} \geqslant U^{(2)}_{m'+\beta} \end{pmatrix} \tag{4.4}$$

$$(\beta = 1, \ldots, m')$$

An example of a diagram of such a function is shown in the figure (with $m = m'$). We include $u_{m'+\beta}$ ($\beta = 1, \ldots, m'$) into the number of control parameters, i.e. we consider that $m = 2m'$, and we establish the relations

$$\psi_k = u_k - \chi_k(u_{m'+k}) = 0 \qquad (k = 1, \ldots, m') \tag{4.5}$$

The optimization problem has now exactly the same formulation as described in Section 1. With the aid of the functions χ_k and the conditions (4.5) we are able to get rid of the limitations (4.2), and thus to shift from the closed region of variation of the parameters $u_1, \ldots, u_{m'}$ to the open region of variation of the parameters $u_1, \ldots, u_{m'}, u_{m'+1}, \ldots, u_m$.

Equations (4.1) and (4.5) can be written in the matrix form

$$\dot{x} = bx + hu_{\rm I}, \qquad \psi = u_{\rm I} - \chi(u_{\rm II}) = 0 \tag{4.6}$$

Here x, ψ, $u = \{u_{\rm I} \mid u_{\rm II}\}$ have the meaning explained above (see (3.13) and (3.14)), and

$$u_{\rm I} = \{u_1, \ldots, u_{m'}\}, \qquad u_{\rm II} = \{u_{m'+1}, \ldots, u_m\} \tag{4.7}$$

are the submatrices of the column matrix u [5]. The symbols b and h denote a square matrix of the order n and a rectangular matrix $n \times m'$

$$b = \begin{Vmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots & b_{nn} \end{Vmatrix}, \qquad h = \begin{Vmatrix} h_{11} & \ldots & h_{1m'} \\ \ldots & \ldots & \ldots \\ h_{n1} & \ldots & h_{nm'} \end{Vmatrix} \tag{4.8}$$

With the aid of the relations (3.14) we construct the equation

$$\dot{\lambda} + b'\lambda = 0 \tag{4.9}$$

which determines the column λ, and we write the matrix of the differential operators $\partial/\partial u$ in the form

$$\frac{\partial}{\partial u} = \left\{\frac{\partial}{\partial u_{\rm I}} \,\vdots\, \frac{\partial}{\partial u_{\rm II}}\right\} = \left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{m'}} \,\vdots\, \frac{\partial}{\partial u_{m'+1}}, \ldots, \frac{\partial}{\partial u_m}\right\} \tag{4.10}$$

On the basis of the equality (3.16) we have

$$\frac{\partial}{\partial u_{\rm I}}[x'b + u_{\rm I}'h']\lambda + \frac{\partial}{\partial u_{\rm I}}[u_{\rm I}' - \chi'(u_{\rm II})]\mu = 0, \qquad \frac{\partial}{\partial u_{\rm II}}\chi'(u_{\rm II})\mu = 0 \tag{4.11}$$

or

$$h'\lambda + \mu = 0 \qquad \frac{\partial \chi'(u_{\rm II})}{\partial u_{\rm II}}\mu = 0 \tag{4.12}$$

where the matrix

$$\frac{\partial \chi'(u_{\text{II}})}{\partial u_{\text{II}}} = \left\| \begin{matrix} \frac{d\chi_1}{du_{m'+1}} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{d\chi_{m'}}{du_{2m'}} \end{matrix} \right\| \qquad (4.13)$$

is diagonal.

The solution of Equation (4.6) has the following form [9]:

$$x = M(t - t_0)x^0 + \int_{t_0}^{t} M(t-\tau) hu_{\text{I}}(\tau)\, d\tau \qquad (M(t) = e^{bt}) \qquad (4.14)$$

A similar expression

$$\lambda(t) = M'(T - t)\lambda(T) \qquad (4.15)$$

gives the solution of Equation (4.9), satisfying the boundary condition for $t = T$. Substituting it into the relation (4.12), we obtain

$$h'M'(T - t)\lambda(T) + \mu = 0$$

Hence we find*

$$\mu = -h'M'(T - t)\lambda(T) \not\equiv 0 \text{ for } \lambda(T) \neq 0 \qquad (4.16)$$

where all the elements of the column μ are different from zero.

Now, on the basis of the equalities (4.12) and (4.13), we obtain

$$\mu_k(t)\frac{d\chi_k}{du_{m'+k}} = 0 \quad \text{or} \quad \frac{d\chi_k}{du_{m'+k}} = 0 \quad (k = 1, \ldots, m') \qquad (4.17)$$

since neither one of $\mu_k(t)$ is identically equal to zero: $\mu_k(t) \not\equiv 0$ ($k = 1, \ldots, m'$).

Consequently, optimum processes in linear systems have an important property: the control parameters in such processes assume only limit values

$$u_k = U_k^{(1)}, \quad \text{or} \quad u_k = U_k^{(2)} \qquad (k = 1, \ldots, m') \qquad (4.18)$$

* We assume that Equation (4.6) is not degenerate [6].

Let us note that the solution (4.14) of Equation (4.16) for $u_I = U =$ const assumes the form

$$x(t) = M(t - t_0)\, x^\circ + N(t - t_0)\, hU \qquad \left(N(t) = \int_0 M(\tau)\, d\tau\right) \tag{4.19}$$

We shall investigate now the continuity condition (3.5) of the function H which, in the case being considered, has the form

$$H_\lambda = \lambda' bx + \lambda' h u_I$$

Only its second term

$$\lambda' h u_I = \sum_{k=1}^{m'} (\lambda' h_k)\, u_k$$

may be discontinuous.

Here h_k denotes the kth column of the matrix h. Therefore, the discontinuities of the control parameters $u_k(t)$ may exist only for $t = t^*$ where the function $\lambda'(t)h_k$ is equal to zero

$$\lambda'(t^*)\, h_k = 0 \tag{4.20}$$

and only the parameter $u_k(t)$ becomes discontinuous unless, obviously, any other column h_α satisfies an equation of the type (4.20) for $t = t^*$.

The results obtained allow for proof of a known [10] theorem of n intervals. For this purpose it is necessary to consider Equations (4.1) with one parameter $u_1 = u$ under the assumption that the principal values of the matrix b are all real. The diagram of the function $\lambda'(t)h_1 = \lambda'(t)h$ then has not more than $n - 1$ intersections with the t-axis on an arbitrary finite interval of time. Consequently, on the interval $t_0 \leqslant t \leqslant T$ the parameter $u(t)$ cannot have more than $n - 1$ discontinuities, and the total interval is divided into n subintervals in which the parameter $u(t)$ assumes either one of its limit values $U^{(1)}$ and $U^{(2)}$.

Repeating similar arguments for systems with m' parameters $u_k(t)$ $(k = 1, \ldots, m')$, we obtain a generalization of this theorem. For optimum processes in such systems, if their characteristic equations have real roots only, the interval $t_0 \leqslant t \leqslant T$ is divided into $m'n$ subintervals in which each of the control parameters assumes one of its limit values $U_k^{(1)}$ or $U_k^{(2)}$.

Let us note that the results obtained are valid for an arbitrary form of the functional J. Some of them can be extended over nonlinear systems [8].

Here, as well as in Section 5, only results which are valid for linear systems are presented. A complete solution of the optimum problem with respect to high-speed performance is given in [6].

5. Examples. We shall consider the problem of optimum transient time for the linear system

$$\dot{x} = bx + hu \tag{5.1}$$

with one control parameter u. In this equation, x and h are one-column matrices of the order n, and b is a square matrix of the same order n. The functional I is to be taken in the form $J = T$, while the conditions (1.4) may be expressed as

$$\Phi = x(T) - x^T = 0 \tag{5.2}$$

The initial conditions for x are given by the relations (3.17). The boundary conditions for the functions $\lambda_s(t)$ may be written in the form of one matrix relation $\lambda(T) = -\rho$. This last relation and the equalities (4.17) give

$$\lambda(t) = -M'(T-t)\rho \tag{5.3}$$

and thus in order to determine the instants of time $t_1, \ldots, t_{q-1}$ corresponding to the switches of the control $u(t)$, we have the relation

$$\lambda'(t_i)h = -\rho' M(T-t_i)h = 0 \tag{5.4}$$

The total interval $t_0 \leqslant t \leqslant T$ splits into the subintervals $t_0 \leqslant t \leqslant t_1$, $t_1 \leqslant t \leqslant t_2$, $\ldots$, $t_{q-1} \leqslant t \leqslant t_q = T$, and for each of them a solution can be constructed

$$t_i \leqslant t \leqslant t_{i+1}, \qquad x(t) = M(t-t_i)x_i + N(t-t_i)hU$$

where the notation $x_i = x(t_i)$ is used, and it is assumed that $U =$ const. At the end of ith interval we have

$$x_{i+1} = x(t_{i+1}) = M(t_{i+1}-t_i)x_i + N(t_{i+1}-t_i)hU \tag{5.5}$$

To be specific, we assume that on the first subinterval $t_0 \leqslant t \leqslant t_1$ the control parameter $u(t) = U_1$. On the second interval it is equal to U_2, on the third it is again equal to U_1, and so on. Writing the equalities of the type (5.5) for each subinterval and eliminating x_i, $i = 1, \ldots, q-1$, we obtain the following formula for an even $q = 2q'$:

$$x_q = x_{2q'} = x(T) = M(T-t_0)x^\circ +$$
$$+\sum_{i=1}^{q'} M(T-t_{2i-1})N(t_{2i-1}-t_{2i-2})hU_1 + \sum_{i=1}^{q'} M(T-t_{2i})N(t_{2i}-t_{2i-1})hU_2 = x^T \tag{5.6}$$

and for an odd $q = 2q' + 1$

$$x_q = x_{2q'+1} = x(T) = M(T - t_0)\,x^\circ + \tag{5.7}$$
$$+ \sum_{i=1}^{q'+1} M(T - t_{2i-1})\,N(t_{2i-1} - t_{2i-2})\,hU_1 + \sum_{i=1}^{q'} M(T - t_{2i})\,N(t_{2i} - t_{2i-1})\,hU_2 = x^T$$

Similar expressions could be written for the case of $u = U_2$ on the first subinterval. They follow from Expressions (5.6) and (5.7) if U_1 is replaced by U_2 and vice-versa. If there is no principal value of the matrix b equal to zero, the matrix $N(t)$ may be represented in the form

$$N(t) = b^{-1}[M(t) - I]$$

where I is the unit matrix. Thus, instead, for instance, of Expression (5.6) we have

$$x(T) = x_{2q'} = M(T - t_0)\,x^\circ + b^{-1}M(T - t_0)\,hU_1 - b^{-1}hU_2 +$$
$$+ \sum_{i=1}^{q'} M(T - t_{2i-1})\,b^{-1}h(U_2 - U_1) + \sum_{i=1}^{q'-1} M(T - t_{2i})\,b^{-1}h(U_1 - U_2) = x^T \tag{5.8}$$

If there are no multiple principal values of the matrix b [5], then

$$b = c\Lambda c^{-1}, \qquad M(t) = e^{bt} = ce^{\Lambda t}c^{-1}$$

where Λ is the diagonal matrix of the principal values λ_i of the matrix b. Expression (5.8) may be written in the form

$$c^{-1}x(T) = e^{\Lambda(T-t_0)}z^\circ + e^{\Lambda(T-t_0)}\Lambda^{-1}h^\circ U_1 - \Lambda^{-1}h^\circ U_2 +$$
$$+ \sum_{i=1}^{q'} e^{\Lambda(T-t_{2i-1})}\Lambda^{-1}h^\circ(U_2 - U_1) + \sum_{i=1}^{q'-1} e^{\Lambda(T-t_{2i})}\Lambda^{-1}h^\circ(U_1 - U_2) = z^T$$

Here

$$z^\circ = c^{-1}x^\circ, \qquad z^T = c^{-1}x^T, \qquad h^\circ = c^{-1}h$$

The same result in scalar form is

$$e^{\lambda_j(T-t_0)}z_j^\circ + e^{\lambda_j(T-t_0)}\frac{h_j^\circ}{\lambda_j}U_1 - \frac{h_j^\circ}{\lambda_j}U_2 +$$
$$+ \sum_{i=1}^{q'} \frac{h_j^\circ}{\lambda_j} e^{\lambda_j(T-t_{2i-1})}(U_2 - U_1) + \sum_{i=1}^{q'-1} \frac{h_j^\circ}{\lambda_j} e^{\lambda_j(T-t_{2i})}(U_1 - U_2) = z_j^T$$

Here z_j°, z_j^T, and h_j° denote the jth elements of the respective columns. In a similar way the remaining relations may be transformed into scalar form.

In order to solve the problem of optimum transient time, it is necessary to select the values ρ_i of the column ρ, which determine the instants of switching the control $u(t)$, in such a way that for $t = T$ the relations of the type (5.6) or (5.7) are satisfied. But in some cases one can avoid this lengthy process of calculation and limit oneself to the solution of Equation (5.6) or (5.7).

To clarify this, let us consider a simple problem of optimum transient time with the condition

$$U_1 \leqslant \ddot{x} \leqslant U_2 \tag{5.9}$$

It can be reduced to the problem discussed above by introducing the following notations:

$$x_1 = x, \quad x_2 = \dot{x} \tag{5.10}$$

and dealing with the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u \qquad (U_1 \leqslant u(t) \leqslant U_2) \tag{5.11}$$

The matrices b, $M(t)$, and $N(t)$ are now of the form

$$b = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \qquad M(t) = \begin{Vmatrix} 1 & t \\ 0 & 1 \end{Vmatrix}, \qquad N(t) = \begin{Vmatrix} t & \frac{1}{2}t^2 \\ 0 & t \end{Vmatrix} \tag{5.12}$$

Substituting them into the equation obtained from (5.6) with $t_0 = 0$ and $q' = 1$

$$x_2 = M(T)\,x^\circ + M(T - t_1)\,N(t_1)\,h\,U_1 + N(T - t_1)\,h\,U_2 = x^T$$

we obtain two scalar equations

$$\begin{gathered} x_1^\circ + Tx_2^\circ + \frac{1}{2}T^2U_1 + \frac{1}{2}(T - t_1)^2(U_2 - U_1) = x_1^T \\ x_2^\circ + TU_1 + (T - t_1)(U_2 - U_1) = x_2^T \end{gathered} \tag{5.13}$$

Their solution has the form

$$T = -\left(\frac{x_2^\circ}{U_1} - \frac{x_2^T}{U_2}\right) \mp \left(\frac{1}{U_1^2}\left(1 - \frac{U_1}{U_2}\right)\left[(x_2^\circ)^2 - \frac{U_1}{U_2}(x_2^T)^2 - 2U_1(x_1^\circ - x_1^T)\right]\right)^{\frac{1}{2}} \tag{5.14}$$

$$T - t_1 = \frac{x_2^T}{U_2} \mp \frac{1}{U_2 - U_1}\left(\frac{1}{U_1^2}\left(1 - \frac{U_1}{U_2}\right)\left[(x_2^\circ)^2 - \frac{U_1}{U_2}(x_2^T)^2 - 2U_1(x_1^\circ - x_1^T)\right]\right)^{\frac{1}{2}} \tag{5.15}$$

From the values of T given by (5.14) the smaller one should be taken into account. Thus, the problem of synthesis of an optimum system may be solved with Expressions (5.14) and (5.15). The results are well known [10] and they will not be repeated here. In this simple case, it was possible to solve the optimization problem without the use of the relations (5.4).

We shall consider now the optimization problem of the linear system (4.1) with the functional

$$J = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i (T) x_k (T) \tag{5.16}$$

being the definite quadratic form of the coordinates $x_s(T)$ at the fixed time $t = T$. There are no limitations imposed on their values. In this case

$$\lambda (T) = - \frac{\partial}{\partial x (T)} J = - a x (T) \tag{5.17}$$

and repeating the calculations described above we obtain the following equation:

$$x' (T) a M (T - t_i) h = 0 \tag{5.18}$$

which determines the instants of time t_i of switching the control $u(t)$. Substituting $x(T)$ from, for instance, the relation (5.6) into (5.18), we obtain

$$\Big\{x^{\circ\prime} M' (T - t_0) + U_1 \sum_{i=1}^{q'} h' N' (t_{2i-1} - t_{2i-2}) M' (T - t_{2i-1}) + + U_2 \sum_{i=1}^{q'} h' N' (t_{2i} - t_{2i-1}) M' (T - t_{2i})\Big\} a M (T - t_j) h = 0 \qquad (j = 1, \ldots, 2q' - 1) \tag{5.19}$$

In a similar way the equations corresponding to odd $q = lq' + 1$ (5.7) or the other sequence of switching controls may be obtained.

In the preceding example of the system (5.11) we had $q' = 1$. From (5.19) we obtain one equation

$$\frac{1}{2} a_{11} (U_2 - U_1) (T - t_1)^3 + \frac{3}{2} a_{12} (U_2 - U_1) (T - t_1)^2 + + \left[\left(x_1^\circ + T x_2^\circ + \frac{1}{2} T^2 U_1\right) a_{11} + (x_2^\circ + T U_1) a_{12}\right] (T - t_1) + + a_{12} \left(x_1^\circ + T x_2^\circ + \frac{1}{2} T^2 U_1\right) + a_{22} (x_2^\circ + T U_1) = 0 \tag{5.20}$$

which determines the instant of time of switching the control $u(t)$.

If the square of the coordinate $x_1^2(T) = x^2(T)$ is minimized, then the coefficients $a_{12} = a_{22} = 0$ and we have the relation

$$(T - t_1)^2 = \frac{1}{U_1 - U_2} (2x_1^\circ + 2T x_2^\circ + T^2 U_1) \tag{5.21}$$

from which the value of t_1 can be easily determined.

In conclusion, the author wishes to express his gratitude to A.I.Lur'e

for his help in this work.

BIBLIOGRAPHY

1. Bliss, G.A., *Lektsii po variatsionnomu ischisleniiu (Lectures on the Calculus of Variations)*. (Russian translation.) IIL, 1950.

2. Giunter, N.M., *Kurs variatsionnogo ischisleniia (Course of Calculus of Variations)*. OGIZ, GITTL, 1941.

3. Pontriagin, L.S., Optimal'nye protsessy regulirovaniia (Optimum processes of control). *Usp. mat. nauk* Vol. 14, No. 1 (85), 1959.

4. Rozonoer, A.I., Printsip maksimuma L.S. Pontriagina v teorii optimal'nykh sistem (Maximum principle of L.S. Pontriagin in the theory of optimum systems), I, II and III. *Avtomatika i Telemekhanika* Vol.20, Nos. 10-12, 1959.

5. Frazer, R.A., Duncan, W.J. and Collar, A.R., *Teoriia matrits i ee prilozheniia k differentsial'nym uravneniiam i dinamike (Elementary Matrices and Some Applications to Dynamics and Differential Equations)*. (Russian translation.) IIL, 1950.

6. Gamkrelidze, R.V., Teoriia optimal'nykh po bystrodeistviiu protsessov v lineinykh sistemakh (The theory of optimum processes with respect to high-speed performance in linear systems). *Izv. Akad. Nauk SSR, Ser. Math.* Vol. 22, No. 4, 1958.

7. Letov, A.M., Analiticheskoe konstruirovanie reguliatorov (Analytical construction of control systems), I, II and III. *Avtomatika i Telemekhanika* Vol. 21, Nos. 4-6, 1960.

8. Miele, A., General variational theory of the flight paths of rocket-powered aircraft, missiles and satellite carriers. *Astronaut. acta* Vol. 4, No. 4, 1958.

9. Troitskii, V.A., K zadache ob avtokolebaniiakh v sistemakh avtomaticheskogo regulirovaniia s dvumia servomotorami postoiannoi skorosti (On the problem of self-sustained oscillations in systems of automatic control with two servo-motors of constant speed). *PMM* Vol.20, No. 5, 1956.

10. Fel'dbaum, A.A., *Vychislitel'nye usdroistva v avtomaticheskikh sistemakh (Calculating Devices in Automatic Systems)*. Fizmatgiz, 1959.

Translated by M.P.B.